5章　補章

問1

（1） 極座標を考えて，$x = r\cos\theta$，$y = r\sin\theta$とおくと

$$\frac{2x^2y}{x^2+y^2} = \frac{2r^2\cos^2\theta\, r\sin\theta}{r^2}$$

$$= 2r\cos^2\theta\sin\theta \leqq 2r$$

$(x,\ y) \to (0,\ 0)$のとき，$r = \sqrt{x^2+y^2} \to 0$より

$$0 \leqq \lim_{(x,\ y)\to(0,\ 0)} \frac{2x^2y}{x^2+y^2} \leqq \lim_{r\to 0} 2r = 0$$

$$0 \leqq \lim_{(x,\ y)\to(0,\ 0)} \frac{2x^2y}{x^2+y^2} \leqq 0$$

よって，$\displaystyle\lim_{(x,\ y)\to(0,\ 0)} \frac{2x^2y}{x^2+y^2} = \mathbf{0}$

（2）極座標を考えて，$x = r\cos\theta$，$y = r\sin\theta$とおくと

$$\frac{x^4+y^4}{x^2+y^2} = \frac{(x^2+y^2)^2 - 2x^2y^2}{x^2+y^2}$$

$$= x^2+y^2 - \frac{(x^2+y^2)^2 - 2x^2y^2}{x^2+y^2}$$

よって

$$\lim_{(x,\ y)\to(0,\ 0)} \frac{x^4+y^4}{x^2+y^2} = \lim_{(x,\ y)\to(0,\ 0)} \left(x^2+y^2 - 2\cdot\frac{x^2y^2}{x^2+y^2}\right)$$

ここで，$\displaystyle\lim_{(x,\ y)\to(0,\ 0)} (x^2+y^2) = 0$

また

$$\lim_{(x,\ y)\to(0,\ 0)} \left(-2\cdot\frac{x^2y^2}{x^2+y^2}\right) = -2\lim_{(x,\ y)\to(0,\ 0)} \frac{x^2y^2}{x^2+y^2}$$

$\displaystyle\lim_{(x,\ y)\to(0,\ 0)} \frac{x^2y^2}{x^2+y^2}$について

極座標を考えて，$x = r\cos\theta$，$y = r\sin\theta$とおくと

$$\frac{x^2y^2}{x^2+y^2} = \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2}$$

$$= r^2\cos^2\theta\sin^2\theta \leqq r^2$$

$(x,\ y) \to (0,\ 0)$のとき，$r = \sqrt{x^2+y^2} \to 0$より

$$0 \leqq \lim_{(x,\ y)\to(0,\ 0)} \frac{2x^2y^2}{x^2+y^2} \leqq \lim_{r\to 0} r^2 = 0$$

$$0 \leqq \lim_{(x,\ y)\to(0,\ 0)} \frac{2x^2y}{x^2+y^2} \leqq 0$$

よって

$$\lim_{(x,\ y)\to(0,\ 0)} \frac{2x^2y}{x^2+y^2} = 0$$

したがって

$$\lim_{(x,\ y)\to(0,\ 0)} \frac{x^4+y^4}{x^2+y^2} = \lim_{(x,\ y)\to(0,\ 0)} \left(x^2+y^2 - 2\cdot\frac{x^2y^2}{x^2+y^2}\right)$$

$$= 0 - 2\cdot 0 = \mathbf{0}$$

問2

（1）$x = r\cos\theta$，$y = r\sin\theta$とおくと

$$\frac{x^4-y^4}{x^2+y^2} = \frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2}$$

$$= x^2 - y^2$$

$$= r^2\cos^2\theta - r^2\sin^2\theta$$

$$= r^2(\cos^2\theta - \sin^2\theta)$$

$$= r^2\cos 2\theta$$

したがって

$$\lim_{(x,\ y)\to(0,\ 0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{r\to 0} r^2\cos 2\theta = 0 = f(0,\ 0)$$

よって，$f(x,\ y)$は**点$(\mathbf{0},\ \mathbf{0})$において連続である．**

（2）$x = r\cos\theta$，$y = r\sin\theta$とおくと

$$\frac{2xy}{x^2+y^2} = \frac{2r\cos\theta\, r\sin\theta}{r^2} = 2\cos\theta\sin\theta$$

これは，θによっていろいろな値をとる．

よって，極限値$\displaystyle\lim_{(x,\ y)\to(0,\ 0)} f(x,\ y)$が存在せず，

$f(x,\ y)$は**点$(\mathbf{0},\ \mathbf{0})$において連続でない．**

問3

$$\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta}$$

$$= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y} r\cos\theta$$

$$= -r\left(\frac{\partial z}{\partial x}\sin\theta - \frac{\partial z}{\partial y}\cos\theta\right)$$

$$\frac{\partial^2 z}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left\{-r\left(\frac{\partial z}{\partial x}\sin\theta - \frac{\partial z}{\partial y}\cos\theta\right)\right\}$$

$$= -r\left\{\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\cos\theta\right)\right\} \quad ・・・①$$

※記述が長くなるため，部分的に計算する．

$$\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\sin\theta\right)=\frac{\partial}{\partial\theta}\frac{\partial z}{\partial x}\cdot\sin\theta+\frac{\partial z}{\partial x}\frac{\partial}{\partial\theta}\sin\theta$$

$$=\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta}+\frac{\partial^2 z}{\partial x\partial y}\frac{\partial y}{\partial\theta}\right)\sin\theta+\frac{\partial z}{\partial x}\cos\theta$$

$$=\left\{\frac{\partial^2 z}{\partial x^2}(-r\sin\theta)+\frac{\partial^2 z}{\partial x\partial y}r\cos\theta\right\}\sin\theta+\frac{\partial z}{\partial x}\cos\theta$$

$$=-r\left(\frac{\partial^2 z}{\partial x^2}\sin^2\theta-\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta\right)+\frac{\partial z}{\partial x}\cos\theta$$

$$\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\cos\theta\right)=\frac{\partial}{\partial\theta}\frac{\partial z}{\partial y}\cdot\cos\theta+\frac{\partial z}{\partial y}\frac{\partial}{\partial\theta}\cos\theta$$

$$=\left(\frac{\partial^2 z}{\partial y\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta}\right)\cos\theta+\frac{\partial z}{\partial y}(-\sin\theta)$$

$$=\left\{\frac{\partial^2 z}{\partial x\partial y}(-r\sin\theta)+\frac{\partial^2 z}{\partial y^2}r\cos\theta\right\}\cos\theta-\frac{\partial z}{\partial y}\sin\theta$$

$$=-r\left(\frac{\partial^2 z}{\partial x\partial y}\sin\theta\cos\theta-\frac{\partial^2 z}{\partial y^2}\cos^2\theta\right)+\frac{\partial z}{\partial y}\cos\theta$$

よって，①より

$$\frac{\partial^2 z}{\partial\theta^2}=-r\left\{-r\left(\frac{\partial^2 z}{\partial x^2}\sin^2\theta-\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta\right)+\frac{\partial z}{\partial x}\cos\theta\right.$$

$$\left.+r\left(\frac{\partial^2 z}{\partial x\partial y}\sin\theta\cos\theta-\frac{\partial^2 z}{\partial y^2}\cos^2\theta\right)+\frac{\partial z}{\partial y}\cos\theta\right\}$$

$$=r^2\left(\frac{\partial^2 z}{\partial x^2}\sin^2\theta-\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta\right)-r\frac{\partial z}{\partial x}\cos\theta$$

$$-r^2\left(\frac{\partial^2 z}{\partial x\partial y}\sin\theta\cos\theta-\frac{\partial^2 z}{\partial y^2}\cos^2\theta\right)-r\frac{\partial z}{\partial y}\cos\theta$$

$$=r^2\left(\frac{\partial^2 z}{\partial x^2}\sin^2\theta-2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\cos^2\theta\right)$$

$$-r\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\cos\theta\right)$$

これと例題 4 の結果より

$$\text{左辺}=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta$$

$$+\frac{1}{r^2}\left\{r^2\left(\frac{\partial^2 z}{\partial x^2}\sin^2\theta-2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\cos^2\theta\right)\right.$$

$$\left.-r\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\cos\theta\right)\right\}$$

$$+\frac{1}{r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\cos\theta\right)$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta$$

$$+\frac{\partial^2 z}{\partial x^2}\sin^2\theta-2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\cos^2\theta$$

$$-\frac{1}{r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\cos\theta\right)+\frac{1}{r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\cos\theta\right)$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta+\frac{\partial^2 z}{\partial x^2}\sin^2\theta+\frac{\partial^2 z}{\partial y^2}\cos^2\theta$$

$$=\frac{\partial^2 z}{\partial x^2}(\cos^2\theta+\sin^2\theta)+\frac{\partial^2 z}{\partial y^2}(\sin^2\theta+\cos^2\theta)$$

$$=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\text{右辺}$$